Causes of Nonlinearities in Low Order Models of the Real Exchange Rate

By

**Yamin Ahmad, UW-Whitewater
Ming Chien Lo, St. Cloud State University
Olena Mykhaylova, University of Richmond**

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University of Wisconsin – Whitewater
Department of Economics
4th Floor Hyland Hall
800 W. Main Street
Whitewater, WI 53190

Tel: (262) 472 -1361

Causes of nonlinearities in low-order models of the real exchange rate

Yamin Ahmad*

Ming Chien Lo[†]

University of Wisconsin - Whitewater

St. Cloud State University

Olena Mykhaylova[‡]

University of Richmond

Abstract

This paper investigates the extent to which modern DSGE models, which feature local currency pricing, home bias, nontraded goods and incomplete markets, can generate nonlinear real exchange rate dynamics that are consistent with those found in the time series literature using data from the current floating period. Our key findings are as follows. First, if the solution to the DSGE model is approximated to the first order, then linearity tests that utilize univariate autoregressions of the real exchange rate suffer from an omitted variables problem, which leads them to overestimate the true incidence of nonlinearity. Consequently, studies that fail to control for this problem may spuriously find evidence of nonlinearities in the data, despite the fact that the data generating process may be linear. Second, we propose a strategy that can largely eliminate this distortion. Finally, we find that DSGE models solved using higher order approximations are capable of generating true structural nonlinearities in real exchange rates both asymptotically and in short samples.

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* Corresponding author. Department of Economics, 800 W Main Street, Whitewater, WI 53190 Email: ahmady@uww.edu, Homepage: <http://facstaff.uww.edu/ahmady/> Tel: (262) 472 5576

[†] Department of Economics, Stewart Hall 380, St. Cloud State University, 720 Fourth Avenue South, St. Cloud, MN 56301, Email: mclo@stcloudstate.edu

[‡] Department of Economics, Robins School of Business, 28 Westhampton Way, University of Richmond, VA 23173, Email: omykhayl@richmond.edu

1 Introduction

Understanding the behavior of exchange rates has always been an issue of great importance for central bankers and policy makers, more so in today's world characterized by the real and financial linkages that exist between countries. For example, the large and growing international current account imbalances faced by many developing and developed nations are prompting researchers to look for the causes and effects of these phenomena. However, any investigation into the international linkages that exist between countries, be it international capital flows, optimal currency areas or cross-country trade patterns, clearly hinges upon a proper understanding of the dynamics of the real exchange rate (RER). Over the course of the last decade, dynamic stochastic general equilibrium (DSGE) models have become very popular as a tool among both researchers and policy makers, and have been used extensively to study the implications of these international connections on consumer welfare and policy conduct. Yet, all the policy prescriptions derived using the DSGE framework are conditional on the models' ability to replicate the observed facts about real and nominal exchange rates. The contribution of our paper pertains to an evaluation of the DSGE framework with regards to its ability to yield real exchange rate dynamics that are consistent with the data, with a specific focus on the existence of nonlinearities in the simulated time series.

Since the seminal paper by Rogoff (1996), researchers have sought to better understand the dynamics of real exchange rates in order to resolve the purchasing power parity (PPP) puzzle present in empirical work. The open economy literature has taken two distinct approaches to the problem. On the theoretical side, DSGE models were updated with a variety of monetary and real frictions to improve the dynamics of simulated exchange rates. Empirical work, on the other hand, has focused on developing better time-series and panel estimation techniques, in particular suggesting

that real exchange rates may exhibit nonlinearities in their behavior.¹ Our paper aims to bridge these two distinct strands of literature by studying the time-series properties of real exchange rates generated by the several widely used variants of international DSGE models.

To date, the vast majority of the theoretical DSGE literature has primarily focused on studying second moments of the real exchange rate (its variance and autocorrelation, as well as correlations with key macroeconomic indicators).² In evaluating this approach, Chari, Kehoe and McGrattan (2002), henceforth CKM, were one of the first to point out that DSGE models with sticky prices and deviations from the law of one price were unable to replicate the observed volatility and persistence of real exchange rates. Several solutions have been put forth in response to this finding, although to our knowledge, the characterization of the actual dynamics of real exchange rates produced by DSGE models has received almost no attention.

Recent empirical studies, which have focused on the time series properties of real exchange rates, have found evidence of nonlinearities in their behavior. The motivation for this approach arose from the failures of a linear framework: assuming linear dynamics and perfect arbitrage implies that the speed of adjustment is constant at all levels of deviations from PPP. However, several reasons have been put forth which would indicate that perfect arbitrage is not observed in the real world, and consequently nonlinearities would exist in the dynamics of real exchange rates. These reasons range from transactions costs (Michael, Nobay and Peel, 1997; Obstfeld and Taylor, 1997) and heterogeneity of agent's beliefs (Kilian and Taylor, 2003) to misalignments in the foreign exchange market resulting in a lack of coordination (Reitz and Taylor, 2008; Sarno and Taylor, 2001). These papers are among many which have found that the data on real exchange rates can

¹For recent surveys on the literature which highlight the dynamics of real exchange rates characterized as a nonlinear process, see Ahmad and Glosser (2011), Taylor (2006), and Taylor and Taylor (2004). Engel, Mark and West (2007) provide a thorough overview of the developments in both theoretical and empirical literature.

²The two exceptions we are aware of are Corsetti, Dedola and Leduc (2008), who estimate exchange rate pass-through coefficients using simulated data, and Rabanal and Rubio-Ramirez (2012), who study the behavior of simulated real exchange rates at all frequencies.

be parsimoniously characterized as smooth transition autoregressive (STAR) process, or one of its variants. However, these studies lack rigorous theoretical foundations that could help to explain the existence of nonlinearities in real exchange rate behavior.

Drawing on the findings of both of these fields, we build a two-country DSGE model that includes a wide array of frictions commonly used in the theoretical literature to explore whether nonlinearities generated by the model translate into the type of nonlinear real exchange rate dynamics reported in the time series literature. The particular type of nonlinearity we consider is the STAR process, given its success in the empirical literature.

Our main findings are as follows. First, the results—and their interpretation—of the standard nonlinearity tests like that of Teräsvirta (1994) depend critically on the underlying data-generating process (DGP). In large-scale nominal DSGE models, real exchange rate behavior is driven by many state variables, including technology levels, last-period prices, and beginning-of-the-period capital stocks. When we solve the model using first-order approximation (so that the DGP of the simulated RER series is linear in these state variables), linearity tests that express RER as a univariate function of its own lags suffer from an *omitted variables problem*, which consequently leads them to overestimate the incidence of nonlinearity in the RER series. Two additional sources of this overestimation arise from the misspecification of the autoregressive order and from distortions in the size of the test itself. We propose a solution to mitigate the effects of the omitted variables problem and demonstrate that including a subset of easily observable state variables as instruments in the linearity test reduces the incidence of nonlinearity down to the size of the test plus a size distortion error. This, in effect, leads us to (correctly) fail to reject the null hypothesis of a linear RER process when the DGP is truly linear.

When we solve the model to second order—so that the DGP is in fact nonlinear—we find incidences

of nonlinearity above and beyond the size of the test (inclusive of the size distortion error) even when we do include the relevant state variables on the right hand side of the test equation. Therefore, we can meaningfully assert that, when solved to second or higher order, some versions of the nominal DSGE models can produce real exchange rates with nonlinear properties described in many recent empirical papers. The corollary of this finding is that empirical researchers should have a structural model of real exchange rate determination in mind before proceeding with linearity tests, and should include the relevant *linear* state variables in the test equation to control for the omitted variables problem. Then, any nonlinearity detected beyond the size of the test can be interpreted as structurally meaningful.

With this observation in mind, we utilize the solutions that have been put forth in response to the CKM critique to calibrate a large-scale theoretical model to the data and to evaluate the incidence of nonlinearity that is present in the simulated real exchange rate series after controlling for the omitted variables problem. We find that the version of the model with incomplete international financial markets yields the highest incidence of nonlinearities among the four basic specifications we consider, and that the combination of local currency pricing, home bias in consumption and incomplete markets yields the highest incidence of nonlinear dynamics among all the examined convolutions of model extensions. Moreover, the consumption home bias and nontraded goods versions of the model yield real exchange rate dynamics that are consistently characterized as exponential smooth transition autoregressive processes, indicating symmetric adjustment to the PPP norm. In contrast, the local currency price version consistently yields asymmetric adjustment, described by the logistic smooth transition autoregression specification.

Finally, our results are robust to sample size: the omitted variables problem is present (and can be eliminated by including state variables in the linearity test), both asymptotically and in shorter

samples that more closely correspond to the available post-Bretton Woods RER data.

2 The benchmark model

We begin our investigation of real exchange rate nonlinearities by using a simple RBC framework composed of two countries, *Home* (H) and *Foreign* (F), each populated by infinitely lived households of measure one; there is no migration. Households consume a combination of home and foreign goods, and international asset markets are complete.

As a matter of notation, subscripts H and F will refer to a good's country of origin; asterisks will indicate that it is consumed in country F . For example, C_H^* denotes consumption of country H 's good in country F . The two economies have a similar structure; therefore, below we present only the Home country equations. Since the setup of the benchmark model and its extensions (outlined in Section 4) are well documented in the literature, we keep our description of them short. Solutions to the agents' problems described below, as well as the market clearing conditions, are available for download from the authors' websites.

2.1 Firms

A representative firm rents capital K_{t-1} from the domestic households at the rate R_t , hires labor L_t at the rate W_t , and produces the home intermediate good according to

$$Y_{H,t} = Z_t K_{t-1}^\nu L_t^{1-\nu},$$

where $0 < \nu < 1$, and Z_t denotes the level of productivity enjoyed by all home firms at time t . The firm maximizes profits

$$\max_{K_{t-1}, L_t} P_{H,t} Y_{H,t} - P_t (W_t L_t + R_t K_{t-1}),$$

where $P_{H,t}$ and P_t are the prices of the intermediate product and the final good (defined below), respectively, and the firm takes all prices and input costs as given.

The final good, used as the numeraire and available for consumption and investment, is aggregated from the home and foreign intermediate goods according to

$$Y_t = \left[\mu^{\frac{1}{\sigma_{HF}}} [Y_{H,t}]^{\frac{\sigma_{HF}-1}{\sigma_{HF}}} + (1-\mu)^{\frac{1}{\sigma_{HF}}} [Y_{F,t}]^{\frac{\sigma_{HF}-1}{\sigma_{HF}}} \right]^{\frac{\sigma_{HF}}{\sigma_{HF}-1}}, \quad (1)$$

where σ_{HF} measures the elasticity of substitution between home and foreign goods, and μ captures the degree of home bias in consumption. Since the law of one price (LOP) clearly holds in this specification, the prices of the two final goods, which also represent the countries' CPIs, are given by

$$\begin{aligned} P_t &= \left[\mu P_{H,t}^{1-\sigma_{HF}} + (1-\mu) P_{F,t}^{*(1-\sigma_{HF})} \right]^{\frac{1}{1-\sigma_{HF}}} \\ P_t^* &= \left[\mu^* P_{F,t}^{*(1-\sigma_{HF})} + (1-\mu^*) P_{H,t}^{1-\sigma_{HF}} \right]^{\frac{1}{1-\sigma_{HF}}} \end{aligned}$$

We define the real exchange rate as $Q_t \equiv P_t^*/P_t$. The only potential source of real exchange rate fluctuations in the benchmark model is home bias in consumption (μ and/or μ^* greater than 0.5).

2.2 Households

The representative household chooses consumption, work effort, capital investment and a portfolio of assets to maximize its expected lifetime utility

$$U_t = E_t \sum_{j=t}^{\infty} \beta^{j-t} \left\{ \frac{C_j^{1-\Theta}}{1-\Theta} + \chi \frac{[1-L_j]^{1-\Theta}}{1-\Theta} \right\} \quad (2)$$

Households in both countries have access to a complete contingent claims market. Each household faces the following budget constraint:

$$E_t[\Delta_{t,t+1} D_t] + P_t (C_t + I_t) = P_t (W_t L_t + R_t K_{t-1}) + D_{t-1}$$

The first term on the left-hand side is the price of a portfolio of state-contingent bonds traded internationally, and D_{t-1} is the payoff of such portfolio in period t . The household's capital accumulation is given by

$$K_t = (1 - \delta)K_{t-1} + I_t - \frac{1}{2}\psi \left[\frac{I_t}{K_{t-1}} - \delta \right]^2 K_{t-1}$$

2.3 Calibration

Each time period in the model corresponds to one quarter. Since the goal of our paper is to understand how standard DSGE models fare in reproducing the observed nonlinearities in the behavior of real exchange rates, we follow CKM in parameterizing our model. The choice of calibrated and estimated parameters comes from the full model, described in Section 4; we list their values below but defer the discussion of most of these parameters until the later section.

We set $\beta = 0.99$, $\chi = 10$, $\delta = 0.023$, and $\nu = 0.33$, values common in DSGE literature. We set $\mu = \mu^* = 0.8$, and assume that home and foreign intermediate goods are substitutes by letting $\sigma_{HF} = 1.5$. The value of the capital adjustment cost parameter ψ is calibrated to replicate the ratio of standard deviations of consumption to output observed in the U.S. data. We set $\Theta = 5$; the high value of the risk aversion parameter is motivated by the need to replicate the observed high volatility of the real exchange rate, which is a function of relative consumptions when asset markets are complete: $Q_t = \Gamma [C_t/C_t^*]^\Theta$.

Productivity in the two countries evolves according to the following autoregressive process:

$$\begin{bmatrix} \ln Z_t \\ \ln Z_t^* \end{bmatrix} = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.95 \end{bmatrix} \begin{bmatrix} \ln Z_{t-1} \\ \ln Z_{t-1}^* \end{bmatrix} + \begin{bmatrix} \varepsilon_{z,t} \\ \varepsilon_{z,t}^* \end{bmatrix}$$

Following CKM, the technology shock process is described by $Var(\varepsilon_z) = Var(\varepsilon_z^*) = 0.007^2$ and $Cov(\varepsilon_z, \varepsilon_z^*) = 1.2 \times 10^{-5}$.

Solution to the model is found using perturbation methods described in Schmitt-Grohé and Uribe (2004) and Collard and Juillard (2001); computer code is written in Dynare (Collard and Juillard, 2003).

3 Theoretical behavior of real exchange rates

We now explore the theoretical properties of the simulated real exchange rate series generated by the model above, as well as the ability of existing empirical tests to capture its inherent theoretical nonlinearities. In general, we assume that the simulated log real exchange rate q_t may be succinctly captured by a univariate autoregressive process of the form

$$q_t = \lambda_1 q_{t-1} + \dots + \lambda_p q_{t-p} + \varepsilon_t, \quad (3)$$

where ε_t is white noise. The simulated series is stationary if the sum of the autoregressive coefficients, $\Lambda \equiv |\sum_{i=1}^p \lambda_i|$, is less than 1.³

3.1 Estimation of the simulated time series

In addressing the main question of this paper, we focus our attention on the extent to which the simulated real exchange rates exhibit nonlinear dynamics. Several recent papers have argued that a possible resolution to Rogoff's (1996) PPP puzzle is to model the real exchange rate as a nonlinear stationary process. The nonlinearities have taken the form of Markov switching, threshold processes, and variants of smooth transition (STAR) models. Although the nonlinear modeling strategies differ across papers, all such studies aim to show that small permanent deviations from PPP are possible due to frictions, whereas large deviations are quickly corrected. Consequently, real exchange rates

³As a precursor to our analysis, we run unit root tests on the simulated data to verify that the RER series is stationary. In all the trials, where the sample size for each trial is 5000 observations, we are able to reject the null hypothesis of nonstationarity.

exhibit mean reverting behavior only when there is a substantial deviation from the level implied by purchasing power parity.

As a first step to understanding the dynamics of the real exchange rates generated by DSGE models, we limit our attention to the STAR type nonlinearity, given its popularity and success in the literature. The dynamics of the real exchange rate (which can also be thought of as a deviation from PPP) can be described as a STAR process as follows:

$$q_t = \sum_{j=1}^p \lambda_j q_{t-j} + \left(\sum_{j=1}^p \tilde{\lambda}_j q_{t-j} \right) F(q_{t-d}, \kappa) + \varepsilon_t, \quad (4)$$

where $\{q_t\}$ is a globally stationary ergodic process, $\varepsilon_t \sim iid(0, \sigma^2)$, and $F(\cdot)$ represents a transition function from one regime to another and determines the degree of mean reversion. Finally, d and κ represent the delay and speed of adjustment parameters, respectively. According to (4), the dynamics of the real exchange rate are driven by the autoregressive parameters $\{\lambda_1, \dots, \lambda_p\}$ in one regime, and $\{\lambda_1 + \tilde{\lambda}_1, \dots, \lambda_p + \tilde{\lambda}_p\}$ in the other.

A popular approach to characterize this type of nonlinearity involves using an ESTAR (exponential smooth transition) model, where the transition function $F(\cdot)$ takes an exponential form:

$$F(\kappa, q_{t-d}) = 1 - \exp \left[-\kappa (q_{t-d} - \bar{q})^2 \right] \quad (5)$$

The function in (5) possesses the following properties: $F : (0, \infty) \rightarrow [0, 1]$, $F(0) = 0$, $\lim_{x \rightarrow \pm\infty} F(x) = 1$, and the function is symmetric around the threshold parameter, $\bar{q}$. An alternative specification of $F(\cdot)$ is the logistic transition function, which yields a logistic smooth transition autoregressive (LSTAR) model. The LSTAR model captures asymmetric adjustment around the threshold parameter, and in this case, the transition function may be written as:

$$F(\kappa, q_{t-d}) = \frac{1}{1 + \exp[-\kappa (q_{t-d} - \bar{q})]} \quad (6)$$

The ESTAR model has been shown to succinctly capture the dynamics of real exchange rates in the current floating period.⁴

The popularity of the STAR framework arises from its ability to demonstrate how the real exchange rate may move smoothly from one type of regime to another, depending on how far the RER value is from a particular threshold. Small deviations from PPP are considered by the STAR framework to be persistent, whereas large deviations exhibit mean-reverting dynamics. In the ESTAR variant, positive and negative deviations from the threshold are treated symmetrically, whereas the LSTAR specification is indicative of asymmetric adjustment.

Empirical estimation of this class of model proceeds in three steps. Once the order of autoregression, p , has been determined through traditional means (e.g., utilizing information criteria like the AIC/SBC), these steps include: testing for linearity; selecting the value of the delay parameter, d ; and choosing between LSTAR and ESTAR specifications. We outline the basic elements of the linearity test next, given its relevance for the question being asked in our paper, although we refer the reader to Teräsvirta (1994) or Teräsvirta and Anderson (1992) for additional details regarding the overall estimation methodology.

Although there are several ways of defining linearity, we assume that nonlinearity is present if the transition function in (4) does not equal zero. For both the ESTAR and LSTAR specifications outlined in (5) and (6), this test can be captured by the null hypothesis $H_0 : \kappa = 0$. If κ were known, then it would be possible to proceed using classical inference techniques. However, since κ and d are typically not known in practice, then (4) is not identified under the null hypothesis, and hence no consistent estimate of either κ or d exists. This is the essence of the problem outlined by

⁴A small sample of papers that finds that the ESTAR model is able to parsimoniously capture the dynamics of the real exchange rate over the current floating period include Baum et al. (2001), Paya and Peel (2006), Sarantis (1999), and Taylor et al. (2001). We use the dollar-sterling rate during this period as the basis for calibrating our full model in Section 4.

Davies (1977).

To address this issue, Teräsvirta (1994) follows the suggestion proposed in Davies (1977) and keeps the unidentified values fixed when deriving a Lagrange Multiplier (LM)-type test for linearity. The idea behind the test statistic used to test $H_0 : \kappa = 0$ involves taking a third order Taylor expansion of the transition function (4) around $\kappa = 0$. Taking p as given, the researcher estimates the auxiliary regression below for a fixed parameter d :

$$q_t = \alpha_{00} + \sum_{j=1}^p (\alpha_{0j}q_{t-j} + \alpha_{1j}q_{t-j}q_{t-d} + \alpha_{2j}q_{t-j}q_{t-d}^2 + \alpha_{3j}q_{t-j}q_{t-d}^3) + \varepsilon_t, \quad (7)$$

and tests the joint null hypothesis that all the coefficients corresponding to the cross products in (7) are zero:

$$H_{0,LIN} = \alpha_{1j} = \alpha_{2j} = \alpha_{3j} = 0 \quad (j = 1, \dots, p) \quad (8)$$

Nonlinearity is detected if the researcher is able to reject $H_{0,LIN}$.

The test above is also used to determine the delay length, d , by running the linearity test for all plausible values of d and picking the one that minimizes the test's p-value (as suggested by Tsay, 1989). We follow Teräsvirta's (1994) and Teräsvirta and Anderson's (1992) methodology in conducting the linearity tests.

We now examine the correspondence between the DGP of the simulated RER series and the specifications (3) and (4).

3.2 Omitted variables problem

A common approach to solving DSGE models is to use linear methods to approximate the policy and transition functions. More specifically, most algorithms used to solve DSGE models take an n -th order Taylor expansion around the steady state (see, for example, Collard and Juillard, 2001;

Kim et al., 2008; and Schmitt-Grohe and Uribe, 2004). In Dynare, the solution approximated to the first order expresses the current value of the endogenous variables as a function of the previous state of the model and the realization of shocks at the beginning of the current period:

$$\mathbf{u}_t = A_u + B_u \mathbf{x}_{t-1} + C_u \boldsymbol{\varepsilon}_t \quad (9a)$$

$$\mathbf{x}_t = A_x + B_x \mathbf{x}_{t-1} + C_x \boldsymbol{\varepsilon}_t \quad (9b)$$

where $\mathbf{u}_t$ is a $k_u \times 1$ vector of non-predetermined variables (controls); $\mathbf{x}_t$ is a $k_x \times 1$ vector of predetermined endogenous variables (states); $\boldsymbol{\varepsilon}_t$ is a $k_\varepsilon \times 1$ vector of predetermined exogenous variables (shocks); and A_i , B_i and C_i , $i \in \{u, x\}$ are appropriately-dimensioned coefficient matrices.

Given that the focus of our paper is on the real exchange rate, assume, for illustrative purposes and without loss of generality, that $k_u = 1$ so that $u_t = q_t$. We demonstrate in Appendix A that the RER series generated by the process (9) cannot be re-written as a univariate $AR(p)$ equation (3), except for the simple case of $k_x = 1$. Instead, if we were to express q_t as a function of its own lags, we would instead obtain an ARMAX process of the following form:

$$q_t = \phi_0 + \sum_{j=1}^p \phi_j q_{t-j} + \boldsymbol{\varphi}'_x \mathbf{x}_{t-2} + \boldsymbol{\varphi}'_\varepsilon \boldsymbol{\varepsilon}_{t-1} + C_u \boldsymbol{\varepsilon}_t \quad (10)$$

where $\boldsymbol{\varphi}_x$ and $\boldsymbol{\varphi}_\varepsilon$ are functions of the coefficients of A_i, B_i, C_i , $i \in \{u, x\}$ and of the sum of the autoregressive coefficients, $\sum_{j=1}^p \phi_j$. What we may infer from this exercise is that if the theoretical model presented in Section 2 appropriately characterizes the economy, and subject to the accuracy of the model's solution (from the first order Taylor expansion), then (10) represents the optimal characterization of the DGP to a first order approximation. Hence, failure to account for the presence of the lagged state variables on the right hand side of (10) when estimating the DGP would lead to an omitted variables problem. Consequently, estimates of the autoregressive coefficients used to represent the dynamics of the real exchange rate may be biased. Conceivably, a parsimonious nonlinear process may capture the dynamics of the data quite well in the presence of omitted

variables under a linear approximation to the model's equations. A simple approach to resolve this issue would be to include some subset of relevant lagged state variables on the right hand side of the auxiliary equation (7) used in the linearity test. We would then expect to see a reduction in the incidence of nonlinearity down to the size (plus any size distortions) of the linearity test. We investigate this claim in the next section.

Another important point to consider is that if the underlying DGP is nonlinear, no matter how complex, it may still be approximated arbitrarily well by an n -th order Taylor expansion. Dynare has the ability to compute model solutions to second order and third order. Second-order approximation to the model's equations introduces nonlinearities into the dynamics of endogenous variables as follows:

$$\mathbf{u}_t = A_u + B_u \mathbf{x}_{t-1} + C_u \boldsymbol{\varepsilon}_t + D_u (\mathbf{x}_{t-1} \otimes \mathbf{x}_{t-1}) + E_u (\boldsymbol{\varepsilon}_t \otimes \boldsymbol{\varepsilon}_t) + G_u (\mathbf{x}_{t-1} \otimes \boldsymbol{\varepsilon}_t) \quad (11a)$$

$$\mathbf{x}_t = A_x + B_x \mathbf{x}_{t-1} + C_x \boldsymbol{\varepsilon}_t + D_x (\mathbf{x}_{t-1} \otimes \mathbf{x}_{t-1}) + E_x (\boldsymbol{\varepsilon}_t \otimes \boldsymbol{\varepsilon}_t) + G_x (\mathbf{x}_{t-1} \otimes \boldsymbol{\varepsilon}_t) \quad (11b)$$

The matrices A_u and A_x now include a constant capturing the variance of future shocks. Suppose again that $\mathbf{u}_t = q_t$. Analogous to the logic presented above for the linear approximation, if one were able to include all the right hand side variables from (11a), then testing the null of linearity for q_t should yield an incidence of nonlinearity that would equal the size of the test. In this case, however, the result is in some sense meaningless, given that we do know that the true DGP is, in fact, nonlinear. Instead, if we were to estimate the equation for q_t as an autoregressive process including only the linear terms:

$$q_t = \phi_0 + \sum_{j=1}^p \phi_j q_{t-j} + \boldsymbol{\varphi}'_x \mathbf{x}_{t-2} + \boldsymbol{\varphi}'_\varepsilon \boldsymbol{\varepsilon}_{t-1} + C_u \boldsymbol{\varepsilon}_t,$$

then incidences of nonlinearity beyond the size of the test would indicate the presence of the higher order terms in the RER DGP. This is the approach we take later in the paper: when running the linearity tests for higher order approximations, we include the linear lagged state variables as

instruments in the auxiliary equation (7). If any of the higher order terms in (7) are found to be significant, and if we are able to reject the null hypothesis (8), then we take it as evidence of nonlinear behavior in the dynamics of the simulated real exchange rate.

3.3 Results of nonlinearity tests

We now turn our attention to the time series properties of the real exchange rates simulated using the benchmark model outlined in Section 2. We generate 1000 trials (each 5000 observations long) of the (log) real exchange rate and then examine the properties of the simulated data in order to infer the underlying dynamics embedded in the series. It should be noted that since the time series literature examining the dynamics of real exchange rates utilizes raw (unfiltered) data, we proceed to examine the dynamics of the simulated data without HP-filtering it.

The properties of the actual data are summarized in Table 1 for the various G7 countries in our sample. For the 1970Q1–2010Q1 period, we are unable to reject the null hypothesis of a unit root in the real exchange rate for the majority of the countries. Although this might be a point of concern, since the DSGE model above produces stationary real exchange rates, it is important to note that other studies were also unable to reject a unit root in post-Bretton Woods data: see, for example, Lothian and Taylor (1996) and Ahmad and Craighead (2011). This result is unsurprising, given the notoriously low power of unit root tests. In fact, several papers have found that once longer spanning data is utilized, the behavior of the real exchange rate appears to be stationary (Lothian and Taylor, 1996 and 2008; Ahmad and Craighead, 2011).⁵ Moreover, when fitting an autoregressive structure to the real exchange rate, we find that Λ , the sum of the autoregressive coefficients in (3), is less than one in absolute value for all countries, which indicates to us the stationarity of the underlying series. Hence we proceed to utilize the simulated RER series generated from the model,

⁵See Taylor, Peel and Sarno (2001, pg. 1016) for more discussion of this issue.

which are stationary in nature.

It is important to note that the empirical literature that estimates STAR type models typically tests for the presence of a unit root in the data, since several recent papers examining the properties of the linearity tests suggest that the size of Teräsvirta and Anderson (1992) and Teräsvirta (1994) tests can be distorted.⁶ When the test is applied to a linear but nonstationary or highly persistent DGP, the null hypothesis of linearity is rejected too often. Sandberg (2008), for example, finds that the null is rejected 31 percent of the time for a test with 5 percent level of significance. As such, the literature has adopted an approach that tests for the presence of a linear unit root process before proceeding to nonlinearity tests.⁷ We follow this approach for each trial, and find that we are able to reject the unit root null in all the trials where the sample size is 5000 observations long.

We then proceed to test for STAR-type nonlinearity vis-à-vis a linear specification for each trial using the methods suggested by Teräsvirta and Anderson (1992) and Teräsvirta (1994), described in Section 3.1. We set the significance level of the linearity test at 5 percent; rejections of the null hypothesis in (8) indicate the presence of nonlinear dynamics that may be captured as ESTAR/LSTAR type processes. Our goal here is to determine the number of times that nonlinearity is detected, and, conditional on this detection, to compute the number of times that an ESTAR/LSTAR specification is chosen. We do this under two scenarios: in the first, we force the order of autoregression, p , to equal one, corresponding to the case where the researcher knows that the model solution and, therefore, the data generated by Dynare, is based on an AR(1) processes of the form (9). In the second scenario, we determine p using the Schwartz criterion (SBC), representing the case where the researcher does not know anything about the DGP and attempts to infer it from the data itself.

⁶The list includes, but is not limited to, Kiliç (2004) and Sandberg (2008).

⁷Kapetanios, Shin and Snell (2003) and Rothe and Sibbertsen (2006) propose several tests which have higher power than the ADF test when the alternative (a nonlinear but globally stationary DGP) is true.

Our results for the linearity test are reported in panel A of Table 2. When we do not restrict p to equal one, and instead fit it based on the SBC criterion, we find that a maximum autoregressive order of $p = 2$ is selected in all instances. Out of the 1000 trials in the benchmark case, p is chosen to equal one, 989 times; in the remaining cases, p is selected to equal two. Therefore, we do not report any results for $p > 2$.⁸

The nominal size of the linearity test, derived from the Monte Carlo results, is approximately 5 percent, although it is an increasing function of the delay parameter, d . At $d = 4$, linearity is rejected 8.4 percent of the times, which could be due to either the omitted variables problem discussed above, or to the test size distortion (although it is much lower than reported in Sandberg, 2008). We detangle these two mechanisms below, when we add the lagged state variables to the test equation. Consequently, the results in the benchmark version of the model are consistent with what we would expect. Given that home bias is the only source of RER movement, and the model is simulated to the first order, we would not expect to detect any significant deviations from PPP. We find that the simulated data is approximated very well by a linear function, at least to the extent of the size of the linearity tests that were employed.

The right side of Panel A reports the fraction of times that an ESTAR model with symmetric adjustment is selected in those cases where the null hypothesis of linearity is rejected. Based on our findings, we conclude that the RER in the benchmark version of the model is best described by the LSTAR specification, although it is important to keep in mind that this result comes from only 4–8 percent of the trials where linearity was rejected.

An additional point to note from panel A is the very small increase, of 0.1 percent, in the incidence of nonlinearity when we go from a scenario where the autoregressive order is known ($p = 1$) to one

⁸In the full model and the extensions explored later, the number of times p equals one or two differs, although the maximum value of p is never selected to be larger than 2 across all the trials.

where it is inferred from the simulated data ($p = 2$) for values of $d > 1$. Although its magnitude is small, even smaller than the size distortion, its presence indicates that there may be a bias arising from the selection of the autoregressive order. As such, it could be a potential factor that contributes to the incidence of nonlinearity observed in the models that we examine in Section 5.

Finally, panel A demonstrates that the estimated RER dynamics can sometimes be captured by a high order linear process when the model solution is approximated to first order. Thus, when we test for the existence of nonlinear dynamics using the methodology advocated by Teräsvirta's (1994) and Teräsvirta and Anderson's (1992), the presence of the omitted variables problem discussed in Section 3.2 can manifest itself in the form of nonlinearities detected by the linearity tests. This could help to explain the higher incidences of nonlinearities (beyond 5 percent) for values of $d > 1$. A large portion of the empirical literature has estimated the dynamics of real exchange rates as univariate processes. A natural question that arises based on our findings is whether the nonlinear models (like the popular STAR framework) are truly capturing theoretical nonlinearities, or whether they are simply misspecified. We address this question below by incorporating the missing state variables into the linearity tests, and by considering the results of higher order approximations.

3.4 Real exchange rate dynamics as a multivariate process

The solution we consider for mitigating the effects of the omitted variables problem on the incidence of nonlinearity is to include the lagged state variables on the right hand side of (3). Dynare solution to the model of Section 2 includes four state variables: K_{t-1} , Z_{t-1} , and the corresponding foreign quantities. The results of estimating the RER dynamics with these variables included in the test are reported in panel B of Table 2.

We find that the size distortion in the linearity test remains present for larger values of d . When we

force the autoregressive order to equal 1, the inclusion of the lagged state variables either maintains or decreases the incidence of nonlinearity for $d = \{1, 2, 3\}$, just as expected; for $d > 3$, we detect a slight increase in the incidence of nonlinearity. When we do not specify the AR order and instead select it using the SBC criterion, we note a small increase in the incidence of nonlinearity compared to the counterpart values for $p = 2$ in panel A, ranging from 0.1 percent for $d = 2$ to 1.2 percent when $d = 4$. We attribute this increase to the misspecification of the autoregressive order.

Overall, the absence of large incidences of nonlinearity in the benchmark case, above and beyond the size of the linearity test, indicates that a linear specification is appropriate given that the benchmark model's solution was computed to a first order approximation. Nonetheless, our conjecture that the inclusion of lagged states should lower the incidence of nonlinearity is confirmed when the autoregressive order is specified correctly.

3.5 Higher order approximations

As our next step, we examine the results of linearity tests when the theoretical model is solved using higher order approximations. Just as before, we generate 1000 trials of the RER data, each 5000 observations long, based on the second and third order approximations to the model's solution. Again, we find that the RER series is largely linear; the results are reported in panels A and B of Table 2. Panel A, which does not include any state variables, indicates that moving from first to second (and third) order approximation actually results in a very small (0.1 percent) decrease in the incidence of nonlinearity. However, when state variables are included (panel B), we observe a marginal increase in the incidence of nonlinearity (between 0.1 to 0.2 percent) for $p = 1$. When we select the autoregressive order based on the SBC information criterion, we once again see a small reduction in the incidence of nonlinearity in going from first to second or third order.

Moreover, going from second to third order of approximation does not affect the computed incidence of nonlinearity, since the simulated data generated by the third order approximation is equal to that generated by the second order approximation to at least six decimal places. Consequently, we do not continue to investigate the dynamics of the real exchange rate simulated using the third order approximation.⁹

To summarize our results so far, we conclude (albeit based on numerically small differences) that (a) small size distortions exist in the linearity test we employed; (b) inclusion of state variables in the linearity test mitigates the omitted variables problem when the researcher correctly specifies the autoregressive order of the RER series; (c) the misspecification of the autoregressive order can lead to an increase in the detected incidence of nonlinearity; and (d) tests of RER series generated using second or higher order approximations can uncover true structural nonlinearities. The results presented in Section 5 confirm these findings (with much larger magnitudes) for large-scale versions of nominal DSGE models.

4 Real exchange rates in nominal DSGE models

Within the international DSGE framework, real exchange rate is typically determined through the PPP equation as a function of nominal exchange rate and international relative prices. In order to generate dynamic behavior of real exchange rates, we must specify the source(s) of deviations from the purchasing power parity. The recent literature has identified several of these:¹⁰ presence of nontraded goods in the consumer basket (Corsetti, Dedola and Leduc, 2008), home bias in consumer preferences (CKM; Faia and Monacelli, 2008; Steinsson, 2008), sticky prices (CKM;

⁹It has been noted in Kim et al. (2008) that high order approximations of DSGE models often result in explosive solutions. This issue can be solved by "pruning out" the extraneous high-order terms in the Taylor series expansion of the model equations; however, pruning for third order approximations has not yet been implemented in Dynare. While the simple RBC framework of Section 2 does not exhibit explosive solutions, we were unable to solve its large scale extensions (discussed in the next section) to orders higher than two.

¹⁰The following references comprise a non-exhaustive list of papers that have used one or more of these features.

Bergin and Feenstra, 2001; Kollmann, 2001), and local currency pricing (CKM; Corsetti, Dedola and Leduc, 2008; Benigno, 2004; Steinsson, 2008). We add these features one at a time to an otherwise basic two-country DSGE model and study the resulting time-series properties of the simulated real exchange rate. We also consider the case of incomplete international asset markets, in which the uncovered interest parity (UIP) condition does not hold, and nominal (and real) exchange rates become decoupled from the relative interest rate movements.

The choice of shocks included in the model also has non-trivial consequences for the behavior of exchange rates. Monetary policy, which requires the presence of nominal rigidities to affect real variables, must itself be quite persistent to generate the observed half-lives of real exchange rates. On the other hand, several studies found that real shocks, which can include (but are not limited to) government spending, technology, price markups, and the cost of international borrowing, are important ingredients of the simulated real exchange rate behavior (Steinsson, 2008; Adolfson et al., 2007; Christiano et al., 2011). We include all of these disturbances in our model to increase the chances of matching simulated time series to the data.

Finally, to capture the potential differences in the economic capacities of the two countries, we assume that they are populated by infinitely lived households of measure M at home and M^* abroad. Therefore, aggregate and per-capita quantities will be different in equilibrium: aggregate consumption is a multiple of the per-capita value.

4.1 Firms

Each country has a continuum of firms that produce tradable goods indexed by f on the unit interval. At time t , each Home firm hires capital and labor from domestic households to produce

one of the varieties of the domestic good according to:

$$Y_{H,t}(f) = M [K_{T,t-1}(f)]^\nu [Z_{T,t} L_{T,t}(f)]^{1-\nu}$$

We use the subscript T to differentiate between tradable and non-tradable goods (introduced in Section 4.4).

Several papers have recently pointed out the importance of technology specification (levels or growth rate) for the dynamics of DSGE models,¹¹ especially when examining unfiltered simulated variables. To capture this channel, we rewrite the technology process in the vector error correction form as follows

$$\begin{bmatrix} \Delta \ln Z_{T,t} \\ \Delta \ln Z_{T,t}^* \end{bmatrix} = \begin{bmatrix} \ln Z_1 \\ \ln Z_2 \end{bmatrix} + \begin{bmatrix} \gamma \\ -\gamma \end{bmatrix} (\ln Z_{T,t-1} - \ln Z_{T,t-1}^*) + \begin{bmatrix} \varepsilon_{z,t}^T \\ \varepsilon_{z,t}^{T*} \end{bmatrix}$$

All goods varieties are then bundled into a composite home good, available for consumption and investment, using the Dixit-Stiglitz aggregator:

$$Y_{H,t} = \left\{ \int_0^1 [Y_{H,t}(f)]^{\frac{\sigma_{T,t}-1}{\sigma_{T,t}}} df \right\}^{\frac{\sigma_{T,t}}{\sigma_{T,t}-1}},$$

where $\sigma_{T,t} > 1$ is the time-varying price markup. This composite good can be used for public and private consumption or private investment. As in Calvo (1983), firms reset their prices each period with a constant probability $(1 - \omega)$; otherwise, the old prices remain in effect. If a firm f gets to announce a new price in period t , it chooses $\tilde{P}_{H,t}(f)$ to maximize its expected discounted future profits

$$E_t \sum_{j=t}^{\infty} \Omega_{t,j} \omega^{j-t} \left\{ \tilde{P}_{H,t}(f) \left[Y_{H,j}^d(f) + Y_{H,j}^{*d}(f) \right] - TC_j \left(Y_{H,j}^d(f) + Y_{H,j}^{*d}(f) \right) \right\}$$

Here $\Omega_{t,j}$ captures the stochastic discount factor of households, the two terms in the brackets denote the aggregate demands for the home goods bundle from domestic and foreign agents, respectively, and $TC(\cdot)$ represents total costs. Notice that in equilibrium, $Y_{H,j}^d(f) + Y_{H,j}^{*d}(f) = Y_{H,t}(f)$.

¹¹See, for example, Christiano et al. (2011) and Rabanal, Rubio-Ramirez and Tuesta (2011).

4.2 Households

A representative household's expected lifetime utility (2) is updated with the variables $C_{T,t}$ and $L_{T,t}$, where $C_{T,t}$ is aggregated from home and foreign bundles using the CES aggregator (1). Analogously to Section 2, the countries' CPIs are given by

$$P_{T,t} = \left[\mu P_{H,t}^{1-\sigma_{HF}} + (1-\mu) S_t P_{F,t}^{*(1-\sigma_{HF})} \right]^{\frac{1}{1-\sigma_{HF}}} \quad (12a)$$

$$P_{T,t}^* = \left[\mu^* P_{F,t}^{*(1-\sigma_{HF})} + (1-\mu^*) S_t^{-1} P_{H,t}^{(1-\sigma_{HF})} \right]^{\frac{1}{1-\sigma_{HF}}} \quad (12b)$$

Here S_t is the nominal exchange rate, expressed in units of Home currency per one unit of Foreign currency. We redefine the real exchange rate as $Q_t \equiv S_t P_{T,t}^* / P_{T,t}$.

Each household faces the following budget constraint:

$$E_t[\Delta_{t,t+1} D_t] + P_{T,t} [C_{T,t} + I_{T,t} - T_{T,t}] = W_t L_{T,t} + D_{t-1} + R_t K_{T,t-1} + \Pi_{T,t}$$

Households receive transfers $P_{T,t} T_{T,t}$ from their government (which can be negative in the event of lump-sum taxation). $\Pi_{T,t}$ represents per-capita profits of all domestic firms. Finally, the household's capital accumulation is given by

$$K_{T,t} = (1-\delta) K_{T,t-1} + I_{T,t} - \frac{1}{2} \psi \left[\frac{I_{T,t}}{K_{T,t-1}} - (\delta + Z_1 - 1) \right]^2 K_{T,t-1}$$

4.3 The government

The Home monetary policy is a variant of the Taylor rule, in which the short-term nominal interest rate responds (with inertia) to deviations of inflation and output from their targets:

$$i_t = (1-\rho_i) \bar{i} + \rho_i i_{t-1} + (1-\rho_i) [\rho_\pi \pi_t + \rho_y y_t^{gap}] + \varepsilon_{i,t} \quad (13)$$

Here $\bar{i} = \beta^{-1} - 1$ is the steady state level of the interest rate, π_t is the growth rate of CPI, and y_t^{gap} denotes output gap, which we measure as the deviation of output from its steady state level.¹²

The Foreign economy pursues independent monetary policy analogous to rule (13).

Government consumption is described by an autoregressive process

$$\ln G_{T,t} = (1 - \rho_g) \ln \overline{G_T} + \rho_g \ln G_{T,t-1} + \varepsilon_{g,t},$$

where $\varepsilon_{g,t}$ is a white noise process, and $\overline{G_T}$ denotes the steady state level of government spending.

Both governments balance their budgets every period, so $G_{T,t} = T_{T,t}$ and $G_{T,t}^* = T_{T,t}^*$.

4.4 Extensions

The benchmark specification, in which PPP holds at all times, generates virtually no real exchange rate movement and therefore serves as a stepping stone to richer frameworks with more frictions. Below we outline four widely used extensions of the basic DSGE model that break the absolute PPP or the UIP relationships and thus introduce richer dynamics of the real exchange rate. We add them one at a time, and also examine several combinations of the extensions in the next section.

4.4.1 Home bias in consumption

The existence of home bias in consumption has been well documented in the trade literature (for example, in Obstfeld and Rogoff, 2000); to introduce it into our model, we require that $\mu + \mu^* \neq 1$. By allowing for a different composition of Home and Foreign consumption baskets, we decouple movements in the two countries' CPI levels (12). To see the impact of this assumption on the RER dynamics, we substitute (12) into the definition of the real exchange rate and linearize around the

¹²Generally, output gap can be modeled as deviation of current output from either (a) its steady state level; (b) flexible-price level; or (c) last period level.

symmetric steady state to obtain

$$q_t = [\mu^* + \mu - 1] \tau_t,$$

where lowercase letters indicate log deviations from steady state, and $\tau_t \equiv p_{F,t}^* + s_t - p_{H,t}$ represents the terms of trade. As the economies move away from the symmetric case ($\mu = \mu^* = 0.5$ or, more generally, $\mu = 1 - \mu^*$), real exchange rate movements grow in magnitude in response to fluctuations in the terms of trade.

4.4.2 Local currency pricing (LCP)

Empirical evidence points to a rather low degree of pass-through from exchange rates to import prices, which of course is inconsistent with our benchmark assumption that the LOP holds in the individual, as well as the aggregate price level.¹³ The literature has identified several potential causes of this phenomenon: local currency pricing by firms with some degree of monopoly power, the presence of nontraded goods either in the product distribution network or directly in the consumer basket, or price stickiness at the consumer level.

We begin by adding international price discrimination by firms, which may optimally choose to charge different prices, $\tilde{P}_{H,t}(f)$ at home and $\tilde{P}_{H,t}^*(f)$ abroad, to maximize the present discounted value of future profits:

$$E_t \sum_{j=t}^{\infty} \Omega_{t,j} \omega^{j-t} \left\{ \tilde{P}_{H,t}(f) Y_{H,j}^d(f) + S_j \tilde{P}_{H,t}^*(f) Y_{H,j}^{*d}(f) - TC_j \left[Y_{H,j}^d(f) + Y_{H,j}^{*d}(f) \right] \right\}$$

The presence of price rigidities implies that firms have to take into account the entire path of future expected nominal exchange rate S_t when setting foreign prices for their products; thus, the LOP

¹³For example, Engel (1993) finds empirical evidence that the volatility of the price of a good relative to a similar good within a country is lower than the volatility of the price of a good relative to the price of the same good in a different country. Engel and Rogers (1996) find that the "border effect" introduces significant variation in the price of a good sold in different countries.

need not hold. The aggregate price indices of the Home goods at home and abroad are given by

$$P_{H,t}^{1-\sigma_T} = (1-\omega)\tilde{P}_{H,t}^{1-\sigma_T} + \omega P_{H,t-1}^{1-\sigma_T} \quad (14a)$$

$$P_{H,t}^{*(1-\sigma_T)} = (1-\omega)\tilde{P}_{H,t}^{*(1-\sigma_T)} + \omega P_{H,t-1}^{*(1-\sigma_T)} \quad (14b)$$

In addition to the home bias mechanism described above, the real exchange rate now responds to international price differences of the same good. Linearizing the definition of the real exchange rate around the steady state yields

$$q_t = [\mu^* + \mu - 1] \tau_t + \mu \varepsilon_t^H + \mu^* \varepsilon_t^F,$$

where ε_t^H and ε_t^F measure deviations from the LOP that arise due to local currency pricing coupled with nominal rigidities: $\varepsilon_t^F \equiv p_{F,t}^* + s_t - p_{F,t}$ and $\varepsilon_t^H \equiv p_{H,t}^* + s_t - p_{H,t}$.

4.4.3 Nontraded goods

The second way of lowering the degree of pass-through from exchange rates to inflation is to introduce nontraded goods. To do so, we add a continuum of firms, indexed by $n \in [0, 1]$, which, similar to the producers of the tradables above, hire capital and labor from local households to produce one of the varieties of non-tradable goods:

$$Y_{N,t}(n) = M [K_{N,t-1}(n)]^{\nu_N} [Z_{N,t} L_{N,t}(n)]^{1-\nu_N}$$

We assume that there are no correlations in the productivity processes across sectors. All the varieties of nontraded goods are bundled together analogously to the tradable output:

$$Y_{N,t} = \left\{ \int_0^1 [Y_{N,t}(n)]^{\frac{\sigma_{N,t}-1}{\sigma_{N,t}}} dn \right\}^{\frac{\sigma_{N,t}}{\sigma_{N,t}-1}}$$

Each firm in the nontraded sector chooses a price $\tilde{P}_{N,t}(n)$ to maximize profits; this price can be reset each period with a probability $(1-\omega)$. For simplicity and symmetry, we extend the specification

of the technological processes as follows:

$$\Delta \ln Z_{T,t} = \ln Z_1 + \gamma (\ln Z_{T,t-1} - \ln Z_{T,t-1}^*) + \gamma (\ln Z_{T,t-1} - \ln Z_{N,t-1}) + \varepsilon_{z,t}^T \quad (15a)$$

$$\Delta \ln Z_{T,t}^* = \ln Z_2 - \gamma (\ln Z_{T,t-1} - \ln Z_{T,t-1}^*) + \gamma (\ln Z_{T,t-1}^* - \ln Z_{N,t-1}^*) + \varepsilon_{z,t}^{T*} \quad (15b)$$

$$\Delta \ln Z_{N,t} = \ln Z_1 - \gamma (\ln Z_{T,t-1} - \ln Z_{N,t-1}) + \varepsilon_{z,t}^N \quad (15c)$$

$$\Delta \ln Z_{N,t}^* = \ln Z_2 - \gamma (\ln Z_{T,t-1}^* - \ln Z_{N,t-1}^*) + \varepsilon_{z,t}^{N*} \quad (15d)$$

The consumption aggregate in the household's utility function (2) is now given by

$$C_t = \left[\eta^{\frac{1}{\sigma_{NT}}} [C_{N,t}]^{\frac{\sigma_{NT}-1}{\sigma_{NT}}} + (1-\eta)^{\frac{1}{\sigma_{NT}}} [C_{T,t}]^{\frac{\sigma_{NT}-1}{\sigma_{NT}}} \right]^{\frac{\sigma_{NT}}{\sigma_{NT}-1}} \quad (16)$$

Here σ_{NT} denotes the elasticity of substitution between tradables and non-tradables, and $0 < \eta < 1$ determines the relative weight of non-tradables in consumption. Investment used to augment capital stock, as well as government purchases, are now also composed of traded and nontraded goods according to (16). The new CPI level is described by

$$P_t = \left[\eta P_{N,t}^{1-\sigma_{NT}} + (1-\eta) P_{T,t}^{1-\sigma_{NT}} \right]^{\frac{1}{1-\sigma_{NT}}} \quad (17)$$

The definitions of $C_{T,t}$ and $P_{T,t}$ are still given by equations (1) and (12a). Finally, the profits of the nontradable sector are added to the household budget constraint.

It is clear from (17) that, even in the absence of home bias, local currency pricing, or nominal rigidities (in which case the composition and the price of the tradable baskets equalize), relative technological growth between traded and nontraded sectors can, through the Balassa–Samuelson effect, lead to movements in the real exchange rate.

4.4.4 Incomplete markets

While the existence of the internationally traded bond simplifies aggregation by equalizing marginal utilities of wealth in the two countries, it has an uncomfortable implication that the variation in

the relative consumption growth rates is directly linked to real exchange rate movements. By combining the international risk-sharing condition and households' first order conditions, we find that the correlation between the real exchange rate and the growth of relative consumption is 1: $\Delta q_{t+1} = \Theta [\Delta c_{t+1} - \Delta c_{t+1}^*]$, where as it is only 0.07 in the data. This observation has motivated many researchers to examine the incomplete international financial markets specification, which has the potential to lower the theoretical correlation to the empirically observed levels.

We continue assuming (for model tractability) that all consumers can perfectly share risks within a country; additionally, a risk-free bond A_t , issued by the foreign country, can be traded internationally. The bond is denominated in foreign currency and offers nominal interest rate i_t^* . We furthermore assume that when borrowing from abroad, home households must pay a risk premium ζ_t . We follow Christiano et al. (2011) in assuming that ζ_t depends on the home country's position in the international asset market and on the international interest rate differential:

$$\zeta_t = \exp \left\{ \theta_d \left(\frac{NFD_t}{4P_{H,t}Y_{H,t}} - \overline{NFD} \right) - \theta_r (i_t^* - i_t) + \varepsilon_{\zeta,t} \right\}$$

The first term in parentheses captures the deviation of the home consumers' net foreign debt to (annual) GDP ratio from its steady state level, and the parameter θ_d measures the responsiveness of risk premium to changes in this ratio.¹⁴ The risk premium payments are distributed evenly to foreign households. The home household's budget constraint becomes

$$\begin{aligned} E_t[\Delta_{t,t+1}D_t] + S_tA_t + P_{T,t}[C_{T,t} + I_{T,t} - T_{T,t}] = \\ = W_tL_{T,t} + D_{t-1} + [1 + i_{t-1}^*\zeta_{t-1}] S_tA_{t-1} + R_tK_{T,t-1} + \Pi_{T,t} \end{aligned}$$

¹⁴Equilibrium dynamics of a small open economy with incomplete asset markets generally include a random walk component; the transaction cost modification guarantees stationarity of the model. See Schmitt-Grohé and Uribe (2003) for explicit treatment of the problem.

4.5 Calibration

Parameter values used in the full model and its extensions are listed in Table 3. Several of them are quite common in the literature and have been discussed in Section 2; we describe the rest below. Since we use the data on the U.S.-UK real exchange rate as a benchmark for model performance, we set $M = 6$ and $M^* = 1$, and let $\mu = \mu^* = 0.94$.

We follow Corsetti, Dedola and Leduc (2008) and the sources referenced therein to calibrate the nontraded sector. We set the elasticity of substitution between tradables and nontradables, σ_{NT} , to 0.74; assume that the share of nontradables in consumption is 60%, and the share of capital in the production of nontradables is $\nu_N = 0.44$. We let $\sigma_T = \sigma_N = 7$, resulting in a 17 percent markup. Setting $\omega = 0.75$ causes price contracts in both sectors to last for four quarters on average.

As described in CKM, our utility function specification is consistent with the existence of a balanced growth path. We stay close to Rabanal, Rubio-Ramirez and Tuesta (2011) in calibrating the VECM process for technology: $\gamma = -0.007$, $Z_1 = Z_2 = 0.001$, and the standard deviations of innovations $\sigma(\varepsilon_z^T) = \sigma(\varepsilon_z^{T*}) = 0.0105$. For simplicity, we assume that the shocks in the nontraded sector have the same covariance matrix as those in the traded sector.

Based on U.S. data, we set the ratio of government debt to GDP to 20 percent, and let $\rho_g = 0.97$. Standard deviation of the government spending shock is set equal to 0.01. We borrow the values for the monetary policy parameters from CKM. However, since the authors do not report a value for the policy shock (and instead calibrate it to match certain moments of the data), we set its standard deviation to 0.016, a value well within the range of U.S. estimates.¹⁵ As in CKM, we calibrate $Cov(\varepsilon_i, \varepsilon_i^*)$ to match the cross-correlation of U.S. and UK outputs.

¹⁵See, for example, Adolfson et al. (2007), Canzoneri et al. (2006) and Christiano et al. (2011).

We follow Adolfson et al. (2007) in calibrating the process for the time-varying price markup. To this end, we assume

$$\sigma_{j,t} = 0.1\sigma_j + 0.90\sigma_{j,t-1} + \varepsilon_{\sigma,t}^j, \quad j \in \{T, N\}$$

We depart from the authors' estimation somewhat by setting $Var(\varepsilon_{\sigma}^T) = Var(\varepsilon_{\sigma}^N) = 2.6 \times 10^{-4}$, equal to the monetary policy shock described above.¹⁶ We set $\overline{NFD} = 0$, $\theta_r = 1.10$, and $Var(\varepsilon_{\zeta}) = 3.9 \times 10^{-5}$, as in Christiano et al. (2011). We calibrate the elasticity of the risk premium to the level of international debt, ξ_d , to replicate the volatility of current account to GDP ratio observed in the U.S. data.

4.6 Sources of real exchange rate dynamics

At this point, it perhaps useful to summarize the various mechanisms that contribute to the dynamics of the real exchange rate. To this end, equation (18) presents the disaggregated definition of the real exchange rate in the full model:

$$Q_t = S_t \frac{\left\{ \eta P_{N,t}^{1-\sigma_{NT}} + (1-\eta) \left[\mu P_{H,t}^{1-\sigma_{HF}} + (1-\mu) P_{F,t}^{1-\sigma_{HF}} \right]^{\frac{1-\sigma_{NT}}{1-\sigma_{HF}}} \right\}^{\frac{1}{1-\sigma_{NT}}}}{\left\{ \eta^* \left(P_{N,t}^* \right)^{1-\sigma_{NT}} + (1-\eta^*) \left[\mu^* \left(P_{F,t}^* \right)^{1-\sigma_{HF}} + (1-\mu^*) \left(P_{H,t}^* \right)^{1-\sigma_{HF}} \right]^{\frac{1-\sigma_{NT}}{1-\sigma_{HF}}} \right\}^{\frac{1}{1-\sigma_{NT}}}} \quad (18)$$

where the prices of tradables evolve according to (14) and its foreign economy equivalent. The real exchange rate fluctuates due to (a) movements of the nominal exchange rate S_t ; (b) presence of nontraded goods ($\eta > 0$); (c) differences in the composition of the tradable baskets ($\mu \neq 1 - \mu^*$); (d) international price discrimination ($P_{H,t} \neq S_t P_{H,t}^*$); and (e) nominal rigidities ($\omega \neq 0$), which force firms to set prices based in part on expectations about future shocks. Finally, the incomplete markets specification breaks the UIP relationship between real interest rates and exchange rates

¹⁶The model estimated in Adolfson et al. (2007) includes many more frictions and shocks than our framework. If we followed their calibration exactly, price markup movements would dominate our model dynamics.

and also decouples the movements of the latter from the dynamics of relative consumptions.

It is also useful to note that (18) clearly demonstrates the existence of structural nonlinearities in RER dynamics arising from the microfoundations of the DSGE models. In Section 5, we explore whether these micro-level nonlinearities translate into the type of macro-level nonlinearities that can be identified using time series methods. If the answer is yes, then it provides an external validity to both literatures: DSGE framework and RER dynamics represented by time series models.

4.7 Business cycle properties of the models

Table 4 summarizes several relevant moments of the simulated data. In addition to examining the benchmark model and its four extensions, we also study several combinations of the structural primitives. Thus, column 8 refers to the specification of the model that includes both LCP and home bias (we dub this version CKM since Chari, Kehoe and McGrattan, 2002 used it as the starting point in their analysis). Columns 8 and 9 add incomplete markets (CKM_IM) and (separately) nontraded goods (CKM_NT) to the CKM model.

The interaction between nominal rigidities and the presence of home bias in consumption (column 4) produces much more volatile real exchange rates compared to the other basic model extensions, matching the data very closely. We find that the various combination of the basic extensions (CKM, column 8; CKM_IM, column 9; and CKM_NT, column 10) overshoot the real exchange rate volatility by 40–90%. Turning to persistence, we find that all specifications, except the ones involving incomplete asset markets, do a good job in matching the empirical autocorrelation of the real exchange rates. As is often found in DSGE literature, other moments are less satisfactory. For example, all versions of the model understate the persistence of GDP, consumption and investment, and none of the models studied in this paper are able to replicate the empirical correlation between

GDP and the real exchange rate.¹⁷

Overall, we conclude that the DSGE framework (outfitted with certain structural features like local currency pricing or nontraded goods) is capable of replicating the observed real exchange rate volatility and persistence, although more work needs to be done to bring other moments of the models in line with the data.

5 Results of nonlinearity tests in Full Model

5.1 First order approximation

To continue with our exploration of the nonlinearities in the RER dynamics, we simulate the four extensions of the full model, as well as several of their combinations, and report the results Tables 5–7.¹⁸ As before, the computations are based on 1000 trials, each 5000 observations long.

Table 5 reports the results of the linearity tests, which do not include any state variables, performed on the first order approximations to the models’ solutions. Relative to the benchmark case, we note the large increase in the incidences of nonlinearity above the 5 percent nominal size of the test, manifested in rejections of the null hypothesis of linearity as frequent as 99.7 percent (in the case of the LCP model). Among the four extensions, only home bias and nontraded goods exhibit fairly low incidence of nonlinearity, comparable to that of the benchmark model. When we look at the combinations of the extensions, the lowest incidence of nonlinearity is found in the CKM_IM model when the maximum autoregressive order is set equal to 1. For the unrestricted case, where the maximum p was chosen by the SBC criterion to equal 2, CKM_IM shows the lowest incidence of

¹⁷The low autocorrelations found in the simulated data may be influenced by the assumed functional form of the technology process in (15). In the versions of the model without long-run technology growth (like the one used in Backus, Kehoe and Kydland, 1992), the autocorrelations of these three macro indicators lie in the 0.58–0.76 range.

¹⁸The only source of RER dynamics in the full model is the presence of nominal rigidities, with the resulting magnitudes of nonlinearity virtually zero. Therefore, we do not report the results of testing this version.

nonlinearity for $d = \{1, 2\}$, and CKM_NT for $d = \{3, 4\}$. Additionally, we find that the symmetric adjustment captured by the ESTAR model is selected in the majority of cases: it fits the home bias and nontraded goods extensions for all values of the delay parameter, d ; for the LCP and incomplete markets versions of the model, it is selected for $d > 1$.

These large incidences of nonlinearity are, at first glance, puzzling, given that the model has been solved to the first order, and the equations relating the RER series to the state variables and shocks are linear. Our analysis in Section 3 indicates, however, that the detected nonlinearities can be attributed to size distortions in the linearity test, biases arising from misspecification of the autoregressive order, or omitted variables. In some sense, the benchmark case serves as the lower bar on the magnitude of the size distortions arising in the linearity test. Any incidences of nonlinearity above that are due to the latter two issues.

In order to examine the validity of our claims summarized at the end of Section 3, we next include the omitted variables when testing the full model extensions to address the omitted variables problem. However, the number of state variables in the full model is significantly higher than in the baseline specification outlined in Section 2. The version with home bias, local currency pricing and incomplete markets (CKM_IM) has, for example, 29 state variables, not including the current period shocks. In order to save on computational time, we choose a small subset of these to be included as additional explanatory variables in the nonlinearity tests: each country's last period capital stock, GDP, and inflation. Our choice of state variables is influenced by the ease of obtaining their real-world equivalents. Capital stocks and output capture and summarize developments in the real side of the economy. The third state variable is meant to control for inflation persistence that has been documented for the U.S. (see for example, Benati, 2008; Blanchard and Gali, 2007; Cogley and Sbordone, 2008; and Fuhrer and Moore, 1995), as well as to reflect the changes in the

nominal side of the economy.

Table 6 shows the results when these state variables are included in the linearity test to account for the omitted variables problem. We notice immediately the sharp reduction in incidence of nonlinearity detected in the RER series. For low values of d , the incidence of nonlinearity approaches 5 percent, the nominal size of the test, although the distortions that we encountered in the benchmark case remain for larger values of the delay parameter. The largest detected incidences of nonlinearity for $p = 1$ are now found only in the CKM and CKM_NT extensions (11–23 percent), and in the incomplete markets version of the model for $d > 2$ (around 70 percent). If the autoregressive order p is chosen using the SBC criterion, then the incidence of nonlinearity declines to the levels dictated by the size of the linearity test (including any distortions) for the first two versions, and to twice that value for the incomplete markets extension. These remaining nonlinearities can potentially arise from the incomplete set of state variables included in the test. Overall, however, we find ample evidence (succinctly summarized in panel C) that the inclusion of several key macro indications in the RER tests significantly reduces (and often completely eliminates) nonlinearities above and beyond the size of the employed test.

Moreover, we note that the inclusion of the state variables changes the type of nonlinear model selected in many of the cases. Only the home bias and nontraded goods versions are unambiguously described by the ESTAR specification, while the remaining extensions are most often represented by the asymmetric adjustment of the LSTAR model (with the exception of the incomplete markets version, which selects the ESTAR framework for larger values of d when $p = 1$). Some caution is warranted when drawing conclusions from these findings. If the true DGP is nonlinear, then the tests can detect nonlinearities but be misleading in choosing their type (e.g., ESTAR versus LSTAR). Psaradaskis and Spagnolo (2002) assume a Markov regime-switching DGP and report

that nonlinearity tests, though of different types than those in our paper, have high power and are able to detect the inadequacy of linearities in the data. Kiliç (2012) focuses on the different types of nonlinearities (ESTAR and LSTAR). He finds that, in general, when nonlinearities are strong, i.e., κ in (5) or (6) is large, the Taylor approximation used in Terasvirta and Anderson (1992) can be misleading as to the shape of the smooth transition functions. In addition, he finds that higher order approximations can result in a larger distortion when estimating the true nonlinear functions. It is important to keep this point in mind when examining the results for the higher order cases, which we do in the next sub-section.

5.2 Second order approximation

We now investigate the incidence of nonlinearity in the RER series generated using the second order approximation to the model extensions; the results of this exercise are reported in Table 7.¹⁹ As discussed at the end of Section 3.2, we control for the omitted variables problem by including only the *linear* state variables in the auxiliary regression (7). The rejection of the null hypothesis of linearity, after controlling for these additional terms, will then serve as evidence of actual structural nonlinearities in the simulated data. Panel C reports the changes in the incidence of nonlinearity once the six state variables mentioned earlier are included in the test.

As Table 7 reports, the incidences of nonlinearity in the various extensions are larger than the nominal size of the linearity test (plus the size distortion) and, unsurprisingly, than the results reported in Table 6. The incomplete markets and the CKM_IM versions of the model yield the largest incidence of nonlinearity across the board, followed by the nontraded goods extension.

Since we control for the omitted variables problem and simulate the model to second order of

¹⁹Due to space considerations, the results of linearity tests without the lagged state variables (second-order analog of Table 6) are not included in the paper, but are available from the authors upon request.

approximation, we can meaningfully state that the structural micro-level nonlinearities that arise due to the presence of nontraded goods and incomplete markets in large scale open economy macro models translate into macro-level nonlinearities that can be identified by time series methods.

The somewhat low incidences of nonlinearity in other versions of the full model can be explained by noting that if there are only a few observations of the RER in the outer (mean-reverting) regime, and the majority of the observations lie within the inner (persistent) regime, then a nonlinear DGP may be captured by a linear framework due to the low power of the test. This issue has been pointed out by Ahmad (2008) in the context of TAR models, which are a special case of the STAR framework. Consequently, one may consider the numbers reported in Table 7 to be a lower bound on the true extent of the nonlinear behavior of the simulated RER series.

Finally, up to the second order approximation, it appears that the home bias, nontraded goods, incomplete markets and CKM_NT versions of the model largely yield a symmetric adjustment mechanism of the RER series (in the form of an ESTAR specification); the remaining versions in the majority of the cases indicate asymmetric adjustment, described by an LSTAR specification. In light of the findings in Kiliç (2012), the lack of robustness in the choice of the STAR specification is understandable. In selecting between the two frameworks, earlier work by Michael, Nobay and Peel (1997) was more agnostic: they indicated that both LSTAR and ESTAR models were potential fits and used the selection criteria outlined by Teräsvirta (1994). The results of their work indicated that the ESTAR model was a better fit when looking at the interwar period. In the subsequent literature, many studies have simply adopted the ESTAR framework de facto, and typically ignore LSTAR and other asymmetric types when it comes to the study of real exchange rates. Our results here are also relevant, since it would indicate a need to consistently apply the selection criteria outlined by Teräsvirta (1994) in determining the appropriate model, if the cause of the underlying

source of nonlinearity is unknown in the data.

6 Small sample analysis

All the statistics reported in the paper so far have been based on analyzing 1000 artificial time series, each 5000 periods long, and are therefore asymptotic in nature. Below we consider how the results of nonlinearity tests change if we shorten the sample size to 160 quarters, same as in the actual data used in empirical studies of real exchange rates. To this end, we examine the first 160 periods of the simulated time series for each trial that was analyzed earlier. The results of the linearity tests, inclusive of the state variables, from both first and second order approximations are presented in Table 8.

Panel A reports the incidences of nonlinearity across the different versions of the model, and panel B summarizes the relative differences in the incidence of nonlinearity, in percentage terms, between the smaller sample period simulations and the longer ones. We note that the benchmark and home bias versions of the model, under both first and second order approximations, yield larger incidences of nonlinearity than in the longer sample (Tables 6 and 7). In comparison, several of the other extensions of the model now yield lower incidences of nonlinearity. For example, the number of times nonlinearity is detected in both incomplete markets versions of the model is much lower across the board when we reduce the sample length. The nontraded goods specification yields higher incidences of nonlinearity for small values of d under the linearized version of the model, the exact opposite of what we observe when using the second order approximation. Interestingly, the CKM and CKM_NT extensions yield higher incidences of nonlinearity when we do not restrict the order of autoregression under either degree of approximation. Overall, however, we find similar results analyzing the small sample RER series to those reported in the previous section.

With regards to the type of nonlinear model selected (not shown), the home bias and nontraded goods versions consistently continue to select an ESTAR specification. Other versions of the model that also be characterized by the ESTAR framework in at least 50 percent of the trials include the benchmark model, CKM and CKM_NT versions. The remaining extensions are best described by the LSTAR framework. Hence, even in the smaller sample, we find evidence of STAR-type nonlinearity, once the omitted variables problem has been accounted for.

7 Conclusion

The goal of this paper has been to study the dynamics of real exchange rates generated by large-scale DSGE models that incorporate four often-used extensions: local currency pricing, home bias in consumption, nontraded goods, and incomplete markets. More specifically, we consider the extent to which the dynamics of real exchange rates in the open economy DSGE models can be characterized as a nonlinear process, which we estimate using a smooth transition autoregressive approach.

Our first key finding is that the results of nonlinearity tests depend crucially on the underlying data generating process of the real exchange rate. Models solved to first order produce real exchange rates that appear to possess nonlinear behavior when relevant state variables are not included in the linearity test; the detected nonlinearity is therefore the result of the omitted variables problem. We fail to reject the null of linearity when we include lagged output, capital stock and inflation as additional explanatory variables in linearity tests. Second, models solved to second order generate true structural nonlinearities in real exchange rates that can be detected even after including linear state variables on the right hand side of linearity tests. These findings seem to suggest that empirical studies of real exchange rate dynamics must be based on an underlying theoretical model so that

the relevant state variables can be included in the test; otherwise, researchers risk interpreting the omitted variables problem as true nonlinearity in the real exchange rate.

The four extensions we consider represent structural micro-level nonlinearities that need not, *a priori*, generate macro-level nonlinearities that can be detected by time series models. Our results suggest that we can identify some of these types to a certain extent. Specifically, we find that non-traded goods and incomplete markets are the two features of DSGE models that are most likely to generate the nonlinearity of real exchange rates seen in the data. Furthermore, the specifications of the model that do produce identifiable nonlinearities can be described by smooth transition autoregressive (STAR) processes. More specifically, we are able to characterize the nonlinear dynamics of real exchange rates as an ESTAR process for the vast majority of simulations that incorporate home bias, and nontraded goods; for the case of local currency pricing, the dynamics indicate an LSTAR process.

This translation of nonlinearities from structural primitives to the time series processes provides validation, in a sense, of the dynamic representations of time series models despite their lack of “micro-founded” theoretical underpinnings. Moreover, it also demonstrates that certain features of DSGE models allow for a reduced-form representation, in the form of time-series models, that captures the salient properties of the macroeconomic data.

Finally, although the choice of DSGE model specifications examined in this paper was motivated by a desire to bridge the two distinct strands of literature, as well as to better understand the dynamics of real exchange rates, one could certainly suggest a variety of other features that could be considered. Extensions that include different exchange rate regimes, dynamic portfolio choice, and barriers to capital flows, we leave to future work.

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	Canada	France	Germany	Italy	Japan	UK
Autocorrelation	0.968	0.929	0.928	0.936	0.958	0.927
ADF Test	-1.603	-2.436	-2.446	-1.964	-2.721*	-2.647*
PP Test	-1.606	-2.289	-2.342	-1.831	-2.535	-2.728*
λ_1	1.168	0.942	0.9403	0.957	1.21	1.097
Λ	0.965	0.942	0.9403	0.957	0.958	0.917
p	2	1	1	1	2	2
σ_{RER}	0.122	0.163	0.175	0.167	0.254	0.149
LJB Test	3.82	1.94	1.22	1.79	4.63	4.37
Λ_{MUE}	0.998	0.975	0.982	0.9752	0.9829	0.9474
[5% 95%]	[0.954 1.028]	[0.911 1.021]	[0.91 1.022]	[0.932 0.996]	[0.933 1.018]	[0.870 1.011]

Table 1: Time series properties of the real exchange rate for different countries between 1970Q1–2010Q1.

Notes:

(a) the ADF and PP tests report the value of the augmented Dickey-Fuller and Phillips Perron test statistics, respectively; (*) represents significance at the 10 percent level.

(b) MUE represents the median unbiased estimate of Λ .

(c) Figures in square brackets represents the 5 and 95 percentiles for the median unbiased estimate of the persistence parameter, computed using Stock's (1991) method.

Order of Taylor Approximation	Incidence of Nonlinearity				Conditional Probability of ESTAR		
	1	2	3		1	2	3

Panel A: Excluding State Variables

max AR order, p	delay, d						
1	1	0.042	0.042	0.042	0.452	0.476	0.476
	2	0.065	0.064	0.064	0.462	0.469	0.469
	3	0.072	0.071	0.071	0.486	0.493	0.493
	4	0.083	0.082	0.082	0.458	0.463	0.463
2	1	0.042	0.042	0.042	0.452	0.476	0.476
	2	0.066	0.065	0.065	0.455	0.462	0.462
	3	0.073	0.072	0.072	0.479	0.486	0.486
	4	0.084	0.083	0.083	0.452	0.458	0.458

Panel B: Including State Variables

max AR order, p	delay, d						
1	1	0.042	0.043	0.043	0.476	0.465	0.465
	2	0.057	0.058	0.058	0.439	0.431	0.431
	3	0.067	0.069	0.069	0.522	0.493	0.493
	4	0.086	0.088	0.088	0.5	0.466	0.466
2	1	0.051	0.043	0.043	0.431	0.465	0.465
	2	0.067	0.06	0.06	0.433	0.433	0.433
	3	0.077	0.071	0.071	0.481	0.493	0.493
	4	0.096	0.09	0.09	0.458	0.467	0.467

Table 2: Incidence of nonlinearity in the benchmark model.

Notes:

(a) Results are based on 1000 simulations, each 5000 observations long after burning the first 500 observation to avoid the initial value bias.

(b) In panel B, the state variables included in the estimation are the two countries' capital stocks and lagged productivities.

Parameter	Description	Value
β	Household discount factor	0.99
Θ	Relative risk aversion	5
χ	Weight on leisure in the utility function	10
σ_{HF}	Elasticity of substitution between imports and exports	1.5
σ_{NT}	Elasticity of substitution between tradables and nontradables	0.74
μ	Consumption home bias	0.94
η	Weight on non-tradables in consumption	0.6
Z_1, Z_2	Technology growth rates	0.001
γ	Speed of adjustment in cointegrating vector	-0.007
ν	Share of capital in production of tradables	0.33
ν_N	Share of capital in production of nontradables	0.44
δ	Capital depreciation rate	0.023
σ_T, σ_N	Elasticity of substitution between goods varieties	7.0
ω	Price stickiness	0.75
ρ_i	Interest rate inertia in the Taylor rule	0.79
ρ_π	Coefficient on inflation in the Taylor rule	2.15
ρ_y	Coefficient on output gap in the Taylor rule	0.23
θ_r	Elasticity of risk premium wrt interest differential	1.10

Table 3: Parameter values used in the full model and its extensions

	Data	Bench	HBias	LCP	NT	IM	CKM	CKM_IM	CKM_NT
St. dev. GDP	1.70	1.71	1.69	1.71	1.72	1.69	1.70	1.69	1.71
St. dev. relative to GDP									
Consumption	0.86	0.83	0.84	0.85	0.86	0.85	0.83	0.85	0.86
Investment	2.59	2.69	2.51	2.79	2.21	2.56	2.72	2.87	2.26
Real exchange rate	3.14	0.45	3.34	1.29	1.58	0.30	4.59	6.09	5.07
Current account	0.13					0.14		0.97	
Autocorrelations									
GDP	0.89	0.33	0.31	0.35	0.34	0.34	0.32	0.36	0.34
Consumption	0.88	0.26	0.23	0.27	0.29	0.22	0.23	0.23	0.26
Investment	0.91	0.42	0.42	0.42	0.43	0.42	0.42	0.43	0.43
Real exchange rate	0.79	0.70	0.70	0.53	0.72	0.40	0.61	0.51	0.65
Cross-correlations									
U.S.-UK GDP	0.53	0.54	0.51	0.79	0.54	0.59	0.54	0.53	0.52
U.S.-UK Consumption	0.43	0.40	0.52	0.39	0.50	0.57	0.30	0.21	0.39
Real exchange rate and GDP	-0.16	0.16	0.15	0.14	0.14	0.19	0.30	0.55	0.20

Table 4: Business cycle properties of the models. The data refer to the U.S. and UK variables over the 1975Q2–2010Q3 period. All data and simulated series are logged and HP-filtered. Since the path of the current account does not affect any allocations when asset markets are complete, we only report its volatility for the two incomplete markets specifications (IM and CKM_IM).

Maximum AR order, p:		p = 1				p = 2			
Delay, d:		1	2	3	4	1	2	3	4
Panel A: Incidence of Nonlinearity									
Model Version:		0.042	0.065	0.072	0.083	0.042	0.066	0.073	0.084
Benchmark		0.40	0.722	0.966	0.997	0.273	0.326	0.535	0.837
LCP		0.075	0.08	0.084	0.088	0.073	0.08	0.083	0.087
HBias		0.099	0.124	0.135	0.150	0.099	0.123	0.134	0.149
Non Traded		0.344	0.932	0.995	0.999	0.291	0.342	0.766	0.95
Incomplete Markets		0.24	0.317	0.508	0.674	0.132	0.151	0.177	0.235
CKM		0.211	0.242	0.292	0.334	0.124	0.138	0.152	0.174
CKM + Non Traded		0.073	0.142	0.207	0.263	0.072	0.135	0.197	0.254
CKM + Incomplete Markets									
Panel B: Nonlinearity type—P(ESTAR STAR)									
Model Version:		0.452	0.462	0.486	0.458	0.452	0.455	0.479	0.452
Benchmark		0.388	0.614	0.816	0.879	0.465	0.445	0.585	0.754
LCP		0.613	0.613	0.607	0.58	0.589	0.60	0.602	0.575
HBias		0.636	0.677	0.637	0.613	0.646	0.675	0.634	0.611
Non Traded		0.288	0.802	0.894	0.911	0.354	0.351	0.698	0.827
Incomplete Markets		0.529	0.521	0.644	0.721	0.523	0.510	0.475	0.502
CKM		0.559	0.512	0.459	0.458	0.532	0.522	0.487	0.448
CKM + Non Traded		0.411	0.472	0.493	0.487	0.389	0.430	0.447	0.445
CKM + Incomplete Markets									

Table 5: Incidence of nonlinearity in first order approximations without the inclusion of state variables.

Notes:

- (a) Results are based on 1000 simulations, each 5000 observations long after burning the first 500 observation to avoid the initial value bias.
(b) The results from the benchmark model are repeated here for ease of comparison.

Maximum AR order, p:		p = 1				p = 2			
Delay, d:		1	2	3	4	1	2	3	4
Panel A: Incidence of Nonlinearity									
Benchmark		0.042	0.057	0.067	0.086	0.051	0.067	0.077	0.096
LCP		0.062	0.091	0.113	0.114	0.04	0.078	0.102	0.128
HBias		0.062	0.07	0.116	0.135	0.062	0.07	0.115	0.134
Non Traded		0.051	0.09	0.219	0.243	0.051	0.09	0.215	0.240
Incomplete Markets		0.032	0.08	0.701	0.712	0.052	0.078	0.152	0.182
CKM		0.113	0.159	0.166	0.174	0.047	0.059	0.073	0.082
CKM + Non Traded		0.177	0.227	0.236	0.251	0.061	0.075	0.093	0.103
CKM + Incomplete Markets		0.059	0.108	0.157	0.192	0.059	0.107	0.158	0.193
Panel B: Nonlinearity type—P(ESTAR STAR)									
Benchmark		0.476	0.439	0.522	0.50	0.431	0.433	0.481	0.458
LCP		0.403	0.374	0.345	0.438	0.325	0.295	0.343	0.383
HBias		0.613	0.586	0.741	0.719	0.613	0.586	0.730	0.701
Non Traded		0.529	0.633	0.781	0.782	0.549	0.622	0.777	0.779
Incomplete Markets		0.25	0.45	0.906	0.90	0.288	0.282	0.467	0.451
CKM		0.425	0.421	0.422	0.420	0.362	0.322	0.329	0.354
CKM + Non Traded		0.554	0.489	0.462	0.458	0.492	0.440	0.430	0.427
CKM + Incomplete Markets		0.407	0.454	0.433	0.401	0.407	0.430	0.405	0.378
Panel C: Percent Change in Incidence									
Benchmark		0	-12	-7	4	21	2	5	14
LCP		-85	-87	-88	-89	-85	-76	-81	-85
HBias		-17	-13	38	53	-15	-13	39	54
Non Traded		-48	-27	62	62	-48	-27	60	61
Incomplete Markets		-91	-91	-30	-29	-82	-77	-80	-81
CKM		-53	-50	-67	-74	-64	-61	-59	-65
CKM + Non Traded		-16	-6	-19	-25	-51	-46	-39	-41
CKM + Incomplete Markets		-19	-24	-24	-27	-18	-21	-20	-24

Table 6: Incidence of nonlinearity in first order approximation when state variables are included.

Notes: (a) results are based on 1000 simulations, each 5000 observations long after burning the first 500 observation to avoid the initial value bias; (b) the exogenous variables included in the estimation of all versions other than benchmark are the two countries' capital stocks, lagged output and lagged inflation; (c) panel C reports the percent change in incidence of nonlinearity from including lagged state variables.

Maximum AR order, p:		p = 1				p = 2			
Delay, d:		1	2	3	4	1	2	3	4
Panel A: Incidence of Nonlinearity									
Benchmark		0.043	0.058	0.069	0.088	0.043	0.06	0.071	0.09
LCP		0.056	0.086	0.116	0.154	0.053	0.083	0.111	0.144
HBias		0.071	0.077	0.114	0.127	0.069	0.076	0.112	0.124
Non Traded		0.109	0.15	0.258	0.29	0.113	0.151	0.257	0.288
Incomplete Markets		0.459	0.669	0.872	0.901	0.587	0.77	0.872	0.922
CKM		0.288	0.346	0.365	0.38	0.064	0.084	0.098	0.11
CKM + Non Traded		0.168	0.213	0.223	0.239	0.065	0.082	0.094	0.101
CKM + Incomplete Markets		0.224	0.39	0.543	0.647	0.221	0.388	0.533	0.642
Panel B: Nonlinearity type—P(ESTAR STAR)									
Benchmark		0.465	0.431	0.493	0.466	0.465	0.433	0.493	0.467
LCP		0.429	0.337	0.31	0.383	0.321	0.277	0.306	0.34
HBias		0.592	0.571	0.675	0.685	0.580	0.553	0.661	0.669
Non Traded		0.45	0.567	0.705	0.714	0.434	0.543	0.685	0.694
Incomplete Markets		0.551	0.592	0.646	0.626	0.506	0.477	0.474	0.486
CKM		0.226	0.243	0.244	0.239	0.469	0.429	0.367	0.336
CKM + Non Traded		0.595	0.516	0.502	0.49	0.508	0.451	0.426	0.426
CKM + Incomplete Markets		0.174	0.195	0.169	0.167	0.163	0.188	0.169	0.168
Panel C: Percentage Change in Incidence									
Benchmark		2	-9	-3	7	2	-8	-1	8
LCP		-89	-88	-88	-85	-86	-80	-82	-83
HBias		-7	-8	31	43	-7	-8	32	43
Non Traded		4	2	61	66	5	3	61	65
Incomplete Markets		-43	-23	-6	-6	-29	-14	-7	-4
CKM		16	6	-30	-45	-56	-50	-52	-56
CKM + Non Traded		-21	-11	-23	-26	-45	-39	-35	-40
CKM + Incomplete Markets		3	-6	-6	-4	4	-1	-4	-2

Table 7: Incidence of nonlinearity in second order approximation when state variables are included.

Notes: (a) results are based on 1000 simulations, each 5000 observations long after burning the first 500 observation to avoid the initial value bias; (b) the exogenous variables included in the estimation of all versions other than benchmark are the two countries' capital stocks, lagged output and lagged inflation; (c) panel C reports the percent change in incidence of nonlinearity from including lagged state variables.

Maximum AR order, p		First Order Approximation								Second Order Approximation							
		1				2				1				2			
Delay, d		1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4

Panel A: Incidence of nonlinearity

Benchmark	0.073	0.127	0.156	0.183	0.114	0.163	0.194	0.219	0.076	0.126	0.156	0.182	0.079	0.122	0.151	0.177
LCP	0.041	0.07	0.116	0.15	0.036	0.066	0.115	0.149	0.041	0.071	0.111	0.148	0.037	0.068	0.107	0.145
HBias	0.095	0.154	0.208	0.237	0.092	0.147	0.201	0.231	0.087	0.155	0.205	0.225	0.085	0.15	0.197	0.217
Non Traded	0.082	0.154	0.198	0.243	0.086	0.157	0.199	0.243	0.087	0.156	0.193	0.244	0.089	0.153	0.189	0.24
Incomplete Markets (IM)	0.029	0.054	0.115	0.156	0.027	0.054	0.116	0.159	0.076	0.135	0.207	0.26	0.079	0.139	0.21	0.259
CKM	0.086	0.124	0.155	0.178	0.079	0.112	0.14	0.166	0.086	0.136	0.166	0.19	0.082	0.121	0.149	0.175
CKM + Non Traded	0.113	0.152	0.201	0.237	0.099	0.134	0.188	0.226	0.121	0.169	0.214	0.248	0.109	0.154	0.199	0.235
CKM + IM	0.039	0.077	0.122	0.166	0.039	0.076	0.121	0.163	0.056	0.121	0.178	0.231	0.056	0.119	0.176	0.227

Panel B: Percentage change in incidence

Benchmark	74	123	133	113	124	143	152	128	77	117	126	107	84	103	113	97
LCP	-34	-23	3	32	-10	-15	13	16	-27	-17	-4	-4	-30	-18	-4	1
HBias	53	120	79	76	48	110	75	72	23	101	80	77	23	97	76	75
Non Traded	61	71	-10	0	69	74	-7	1	-20	4	-25	-16	-21	1	-26	-17
Incomplete Markets (IM)	-9	-33	-84	-78	-48	-31	-24	-13	-83	-80	-76	-71	-87	-82	-76	-72
CKM	-24	-22	-7	2	68	90	92	102	-70	-61	-55	-50	28	44	52	59
CKM + Non Traded	-36	-33	-15	-6	62	79	102	119	-28	-21	-4	4	68	88	112	133
CKM + IM	-34	-29	-22	-14	-34	-29	-23	-16	-75	-69	-67	-64	-75	-69	-67	-65

Table 8: Incidence of nonlinearity in the small samples when state variables are included.

Notes:

- (a) Simulations are based on 160 observations of simulated data after burning the first 500 observation to avoid the initial value bias.
- (b) The exogenous variables included in the estimation of all versions other than benchmark are the two countries' capital stocks, lagged output and lagged inflation.
- (c) Panel B shows the percentage change in the incidence of nonlinearity relative to the earlier trials with 5000 observations each.

A Converting policy functions into autoregressive processes

The nomenclature used in this appendix pertains only to the derivations presented here, and does not refer to the variables or parameters in the main body of the paper.

A.1 One state, one control

Suppose the linearized (to the first order) model takes the form

$$u_t = b_0 + b_1 x_{t-1} + b_2 \varepsilon_t \quad (19a)$$

$$x_t = a_0 + a_1 x_{t-1} + a_2 \varepsilon_t \quad (19b)$$

where u_t is a (non-predetermined endogenous) control variable; x_t represents a (predetermined endogenous) state variable, and ε_t is an exogenous innovation with mean zero and standard deviation σ^2 drawn independently and identically from an underlying distribution with bounded support. For illustrative purposes, we assume that the same shock affects the state and control variables, and the magnitude of its impact is given by the known parameters a_2 and b_2 , respectively. Such a system adequately represents the output from Dynare. In order to rewrite the system (19) as an autoregressive process for the control variable u_t , we lag the equation (19b) by one period, substitute the resulting expressions into (19a), and collect like terms. We can express u_t as a function of its own lag and the exogenous shock as follows:

$$u_t = [(1 - a_1) b_0 + a_0 b_1] + a_1 u_{t-1} + (a_2 b_1 - a_1 b_2) \varepsilon_{t-1} + b_2 \varepsilon_t$$

As long as the process $\{e_t\}$ is not serially correlated, u_t can be estimated as an ARMA(1,1) process.

A.2 Two states, one control

For more complex systems with more than one state variable, we can demonstrate that the control variable cannot be rewritten as simple univariate autoregressive function. To see this, suppose the model linearized to first order is captured by

$$x_t = a_0 + a_1x_{t-1} + a_2y_{t-1} + a_3\eta_t \quad (20a)$$

$$y_t = b_0 + b_1x_{t-1} + b_2y_{t-1} + b_3\nu_t \quad (20b)$$

$$u_t = c_0 + c_1x_{t-1} + c_2y_{t-1} + c_3\varepsilon_t, \quad (20c)$$

where u_t once again is a control variable, and (x_t, y_t) represent two state variables; however, our argument below applies to models with more than two state variables as well. Here, we allow for the possibility of different shocks affecting the state and control variables. In writing the control variable u_t as a function of its own lags, we follow the algorithm of the previous subsection to re-express the last equation in (20) as follows:

$$\begin{aligned} u_t = & c_0 + c_1 \underbrace{(a_0 + a_1x_{t-2} + a_2y_{t-2} + a_3\eta_{t-1})}_{x_{t-1}} \\ & + c_2 \underbrace{(b_0 + b_1x_{t-2} + b_2y_{t-2} + b_3\nu_{t-1})}_{y_{t-1}} \\ & + c_3\varepsilon_t \end{aligned}$$

We can rewrite the last expression as

$$\begin{aligned} u_t = & [a_0c_1 + b_0c_2 + (1 - \phi)c_0] + \phi[c_0 + c_1x_{t-2} + c_2y_{t-2} + c_3\varepsilon_{t-1}] \\ & + [(a_1 - \phi)c_1 + b_1c_2]x_{t-2} \\ & + [a_2c_1 + (b_2 - \phi)c_2]y_{t-2} \\ & + a_3c_1\eta_{t-1} + b_3c_2\nu_{t-1} - \phi c_3\varepsilon_{t-1} + c_3\varepsilon_t, \end{aligned}$$

where ϕ represents the autoregressive parameter to be estimated from the data. Thus, we can write u_t as an autoregressive process:

$$u_t = \phi_0 + \phi u_{t-1} + \boldsymbol{\varphi}'_1 \mathbf{w}_{t-2} + \boldsymbol{\varphi}'_2 \boldsymbol{\xi}_{t-1} + c_3 \varepsilon_t \quad (21)$$

Here $\phi_0 = a_0 c_1 + b_0 c_2 + (1 - \phi) c_0$, $\boldsymbol{\varphi}_1 = \begin{pmatrix} (a_1 - \phi) c_1 + b_1 c_2 \\ a_2 c_1 + (b_2 - \phi) c_2 \end{pmatrix}$, and $\boldsymbol{\varphi}_2 = (a_3 c_1, b_3 c_2, -\phi c_3)'$. In addition, $\mathbf{w}_{t-2} = (x_{t-2}, y_{t-2})'$ is the vector of lagged state variables, and $\boldsymbol{\xi}_{t-1} = (\eta_{t-1}, \nu_{t-1}, \varepsilon_{t-1})'$ contains the last period shocks. The presence of the term $\mathbf{w}_{t-2}$ on the right hand side of (21) means that it is not possible to express u_t as a simple univariate process. Additionally, we note that (21) is a general solution, and thus the parameter ϕ is not uniquely identified without an independent equation pinning down the relationship between u_t and its lagged value.

In general, we can express (20) as an $AR(p)$ process for u_t as follows:

$$u_t = \phi_0 + \sum_{j=1}^p \phi_j u_{t-j} + \boldsymbol{\varphi}'_w \mathbf{w}_{t-2} + \boldsymbol{\varphi}'_\xi \boldsymbol{\xi}_{t-1} + c_3 \varepsilon_t, \quad (22)$$

where now $\boldsymbol{\varphi}_w = \begin{pmatrix} (a_1 - \sum_j \phi_j) c_1 + b_1 c_2 \\ a_2 c_1 + (b_2 - \sum_j \phi_j) c_2 \end{pmatrix}$ and $\boldsymbol{\varphi}_\xi = (a_3 c_1, b_3 c_2, -\sum_j \phi_j c_3)'$ for $j = 1, \dots, p$. Once again, ϕ_j 's are not identified $\forall j$. Consequently, in the $AR(p)$ case, we would need a set of p independent relationships to pin down the particular solution.

Given that the true DGP may be expressed as (22), a researcher studying the dynamics of u_t by estimating the auxiliary equation

$$u_t = \phi_0 + \sum_{j=1}^p \phi_j u_{t-j} + \varepsilon_t$$

would find biased estimates in the coefficients ϕ_j if she fails to account for the presence of the lagged state variables $\mathbf{w}_{t-2}$ and the vector of predetermined shocks $\boldsymbol{\xi}_{t-1}$. Consequently, univariate regressions estimating a variable whose dynamics are governed by (20) will suffer from an omitted variables problem.